5.11 global stability and Liapunov functions

Wednesday, March 31, 2021 2:52 PM

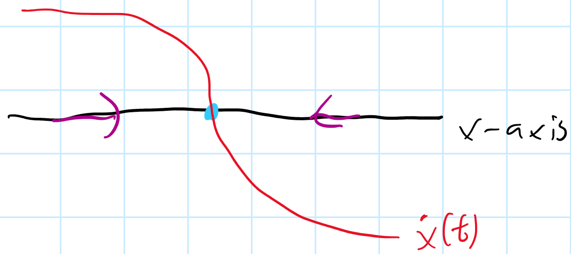
Recall: In Sec. 2.7.2, we had Liapunov exponents for difference equation $x_{t+1} = f(x_t)$ by

$$\lambda(x_0) = \lim_{t\to\infty} \frac{1}{t} \sum_{k=0}^{t-1} \ln|f'(x_k)|$$

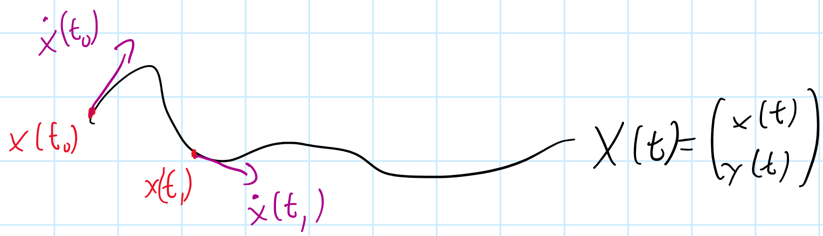
If $\lambda(x_0) > 0$, then there is sensitive dependence on starting pt x_0.

Note: $\lambda(x_0) = \lambda(x_1) = \lambda(x_2) = \cdots = \lambda(x_k)$ because we care about asymp. behavior.

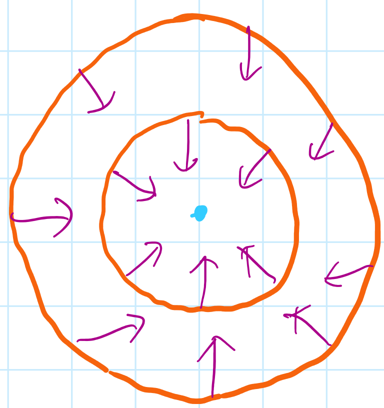
1D



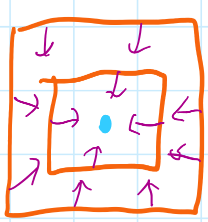
2D



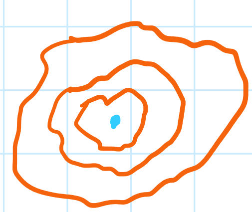
Suppose:



- Construct concentric circles around and equilibrium for every radius r
- If vector field always points inward, then anything in that region will eventually go to the equilibrium
- Can think of concentric circles as the steps the solution takes.



Also works for other shapes



Method of Liapunov

Consider the 2D autonomous system

$$\frac{dx}{dt} = f(x,y) \qquad \frac{dy}{dt} = g(x,y).$$

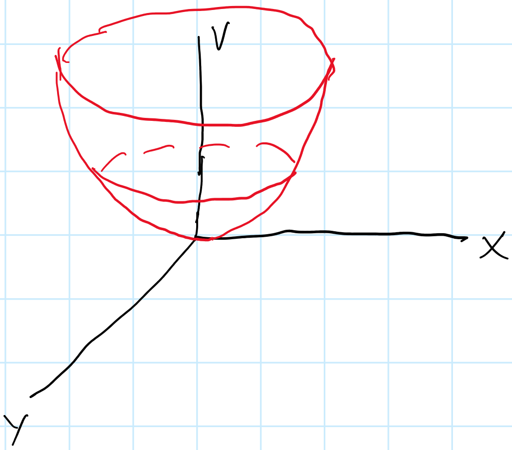
Assume WLOG that the origin is our equilibrium of interest.

Def. 5.10 Let $U \subseteq \mathbb{R}^2$ be open, and $(0,0) \in U$.
A real-valued function $V: U \to \mathbb{R}$ with continuous partial derivatives is **positive definite** on U if:

(1) $V(0,0) = 0$

(2) $V(x,y) > 0 \quad \forall (x,y) \in U$ with $(x,y) \neq (0,0)$.

Ex. $V(x,y) = x^2 + y^2$ is positive def on $\mathbb{R}^2$.



If $x^2 + y^2 = 0$, then $x = 0$, $y = 0$.
And $x^2 + y^2 \geq 0 \quad \forall (x,y) \in \mathbb{R}^2$.

$V(x,y) = x^2 + y^2 - y^3$ is pos def in a region near the x-axis.

$V(x,y) = x + y^2$, $V(x,y) = (x+y)^2$, $V(x,y) = x^2$ are <u>NOT</u> pos def.

Define: The **level sets** of V are $V^{-1}(k) = \{(x,y) \in \mathbb{R}^2 : V(x,y) = k\}$

Note, in order to be pos. def. $V^{-1}(0) = \{(0,0)\}$

$V^{-1}(-a) = \emptyset$ for $a > 0$.

Ex. $V(x,y) = x + y^2$ $\qquad V(x,y) = (x+y)^2$ $\qquad V(x,y) = x^2$

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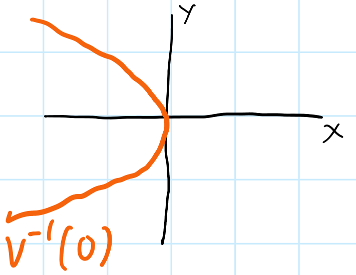
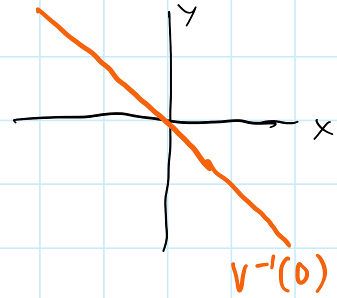
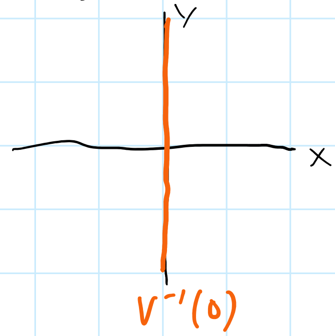


$V(x,y) = (x+y)^2$

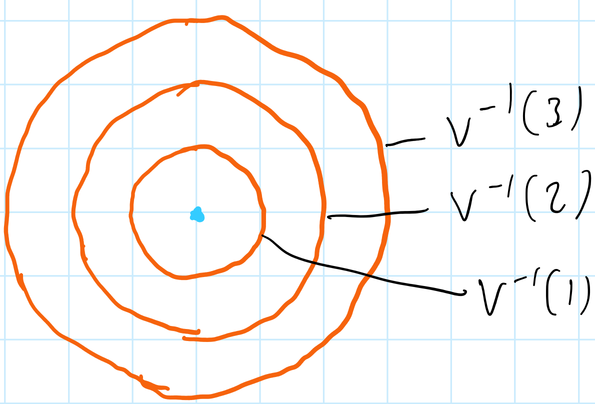


$V(x,y) = x^2$



Ex. $V(x,y) = x^2+y^2$

$V^{-1}(k) = \{(x,y) \mid x^2+y^2 = k\}$



A pos def function's level sets let us construct the regions we wanted.

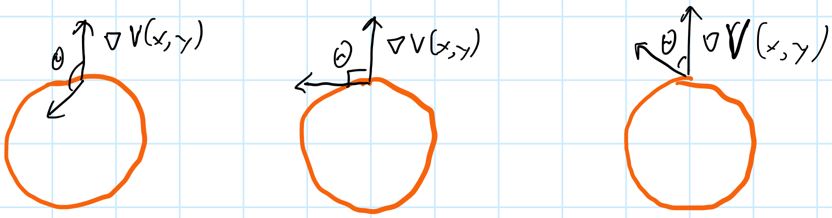
Consider: Let $(x(t), y(t))$ be a solution to our differential system.

Then
$$\frac{dV(x(t),y(t))}{dt} = \frac{\partial V}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial V}{\partial y}\cdot\frac{dy}{dt}$$
$$= \frac{\partial V}{\partial x}\cdot f(x,y) + \frac{\partial V}{\partial y}\cdot g(x,y)$$
$$= \begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix} \underset{\text{dot product}}{\bullet} \underbrace{\begin{bmatrix} \frac{\partial V}{\partial x} \\ \frac{\partial V}{\partial y} \end{bmatrix}}_{\text{gradient } \nabla V(x,y)}$$

$$= \left\| \begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix} \right\| \, \left\| \nabla V(x,y) \right\| \cos\theta,$$

where θ is the angle between (f,g) and $\nabla V(x,y)$.

Note: On a level curve $V^{-1}(k)$ for pos def V, $\nabla V(x,y)$ points outward.



$\cos\theta < 0$ $\quad$ $\cos\theta = 0$ $\quad$ $\cos\theta > 0$

$$\Rightarrow \quad \frac{dV(x,y)}{dt} < 0 \qquad \frac{dV(x,y)}{dt} = 0 \qquad \frac{dV(x,y)}{dt} > 0$$

Thm 5.14 (Liapunov's stability Thm)

Let $(0,0)$ be an equilibrium of the autonomous system $\dot{x} = f(x,y)$, $\dot{y} = g(x,y)$, and let V be a pos. def. function with continuous partial derivatives in a neighborhood U of the origin.

(i) If $\frac{dV(x,y)}{dt} \le 0$, for all $(x,y) \in U - \{(0,0)\}$, then $(0,0)$ is stable and V is called a **Liapunov function**

(ii) If $\frac{dV(x,y)}{dt} < 0$, for all $(x,y) \in U - \{(0,0)\}$, then $(0,0)$ is asymptotically stable on U, and V is called a **strict Liapunov function.**

(iii) If $\frac{dV(x,y)}{dt} > 0$ for all $(x,y) \in U - \{(0,0)\}$, then $(0,0)$ is unstable.

Difficulty: Finding Liapunov functions is hard.

Ex. 5.28 Lotka - Volterra

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$$\frac{dx}{dt} = x(a-y)$$

$$\frac{dy}{dt} = y(-b+x), \qquad a, b > 0.$$

Positive eq $(\bar{x}, \bar{y}) = (b, a)$, which is stable but not asymp. stable.

Consider $V(x,y) = \left(x - b - b \ln \frac{x}{b}\right) + \left(y - a - a \ln \frac{y}{a}\right)$

Can check $V(b, a) = 0$

and $V(x,y) > 0$ for $x, y \in \mathbb{R}^+$ and $(x, y) \neq (a, b)$.

Also,
$$\frac{dV}{dt} = \frac{\partial V}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial V}{\partial y} \cdot \frac{dy}{dt}$$

$$= \left(1 - \frac{b}{x}\right) x(a-y) + \left(1 - \frac{a}{y}\right) y(-b+x)$$

$$= (x-b)(a-y) + (y-a)(x-b) = 0$$

$\Rightarrow$ (b, a) is globally stable in $\mathbb{R}^2_+$.